The vertical gradient of potential density

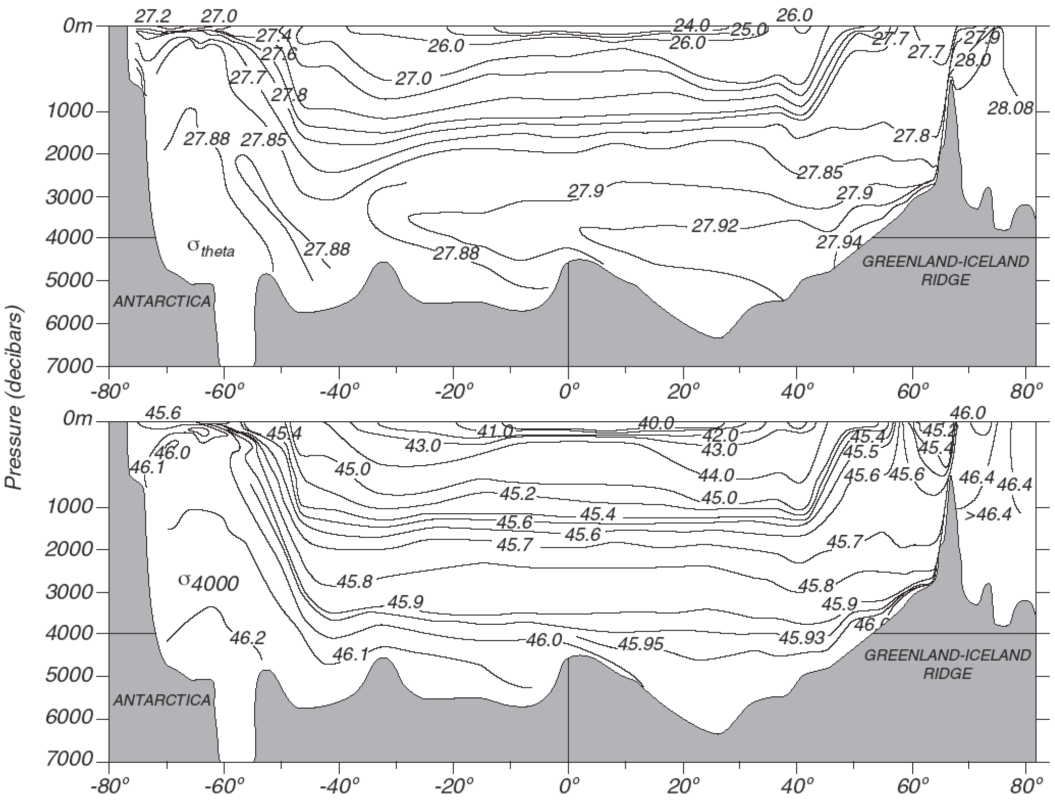


Figure 6.10 Vertical sections of density in the western Atlantic. Note that the depth scale changes at 1000 m depth. **Upper**: σ_Θ, showing an apparent density inversion below 3,000 m. **Lower**: σ_4 showing continuous increase in density with depth. After Lynn and Reid (1968).

The potential density of a seawater sample (S_A, Θ, p), referenced to reference pressure p_r is given by $\rho^\Theta(S_A, \Theta) = \hat{\rho}(S_A, \Theta, p_r)$. The vertical gradient of the natural logarithm of potential density is $\beta^\Theta(p_r)$ times the vertical gradient of Absolute Salinity minus $\alpha^\Theta(p_r)$ times the vertical gradient of Conservative Temperature,

$$\frac{1}{\rho^\Theta}\frac{\partial \rho^\Theta}{\partial z} = \beta^\Theta(p_r) S_{A_z} - \alpha^\Theta(p_r)\Theta_z . \tag{A.26.2}$$

The ratio of this vertical gradient of potential density to the square of the buoyancy frequency is given by (Tutorial exercise)

$$\frac{-g\rho^{-1}\rho_z^\Theta}{N^2} = \frac{\beta^\Theta(p_r)}{\beta^\Theta(p)}\frac{\left[R_\rho/r-1\right]}{\left[R_\rho-1\right]} = \frac{\beta^\Theta(p_r)}{\beta^\Theta(p)}\frac{1}{G^\Theta} \approx \frac{1}{G^\Theta}, \tag{3.20.5}$$

where r is the ratio of the slope on the $S_A - \Theta$ diagram of an isoline of potential density with reference pressure p_r to the slope of a potential density surface with reference pressure p, and is defined by

$$r = \frac{\alpha^\Theta(S_A,\Theta,p)/\beta^\Theta(S_A,\Theta,p)}{\alpha^\Theta(S_A,\Theta,p_r)/\beta^\Theta(S_A,\Theta,p_r)}, \tag{3.17.2}$$

and the "isopycnal temperature gradient ratio" G^Θ is defined by

$$G^\Theta \equiv \frac{\left[R_\rho-1\right]}{\left[R_\rho/r-1\right]} \qquad \text{where} \qquad R_\rho = \frac{\alpha^\Theta\Theta_z}{\beta^\Theta(S_A)_z} \tag{3.17.4}$$

is the ratio of the vertical contribution from Conservative Temperature to that from Absolute Salinity to the static stability N^2 of the water column. The name "isopycnal temperature gradient ratio" is chosen for G^Θ because it can be shown that G^{Θ} is the ratio of the gradient of Conservative Temperature in a potential density surface to that in a neutral tangent plane (Tutorial exercise),

$$\nabla_{\sigma}\Theta \;=\; G^{\Theta}\nabla_{n}\Theta \;. \qquad (3.17.3)$$

The saline contraction coefficient $\beta^{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)$ does not vary very much from a constant value compared with variation of the thermal expansion coefficient $\alpha^{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)$. That is, you make a 10% - 20% error by approximating r as

$$r \;\approx\; \frac{\alpha^{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)}{\alpha^{\Theta}\left(S_{\mathrm{A}},\Theta,p_{\mathrm{r}}\right)} \;. \qquad (3.17.2\_approx)$$

There is never any reason to actually make this approximation in numerical work, rather this approximation can aid in thinking about what causes what in the ocean. [You can check that this is a good approximation by inspection of the red and blue potential density contours on the above $S_{\mathrm{A}}-\Theta$ diagram.]

Also, the slope difference between that of a neutral tangent plane and a potential density surface is given by (Tutorial exercise)

$$\begin{aligned} \nabla_{n}z - \nabla_{\sigma}z \;&=\; \frac{\nabla_{n}\Theta - \nabla_{\sigma}\Theta}{\Theta_{z}} \;=\; \left(1-G^{\Theta}\right)\frac{\nabla_{n}\Theta}{\Theta_{z}} \\ &=\; \frac{R_{\rho}\left[1-r\right]}{\left[R_{\rho}-r\right]}\frac{\nabla_{n}\Theta}{\Theta_{z}} \;=\; \frac{R_{\rho}\left[1-r\right]}{r\left[R_{\rho}-1\right]}\frac{\nabla_{\sigma}\Theta}{\Theta_{z}}. \end{aligned} \qquad (3.18.1)$$

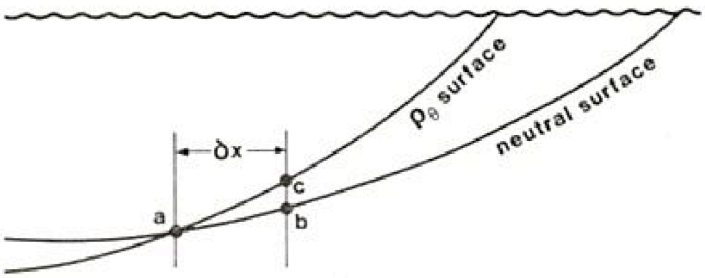


FIG. 1. Sketch of a cross section through the ocean showing a neutral surface and a potential density surface passing through point a. At a horizontal distance δx from point a, a vertical cast cuts the two surfaces at points b and c.

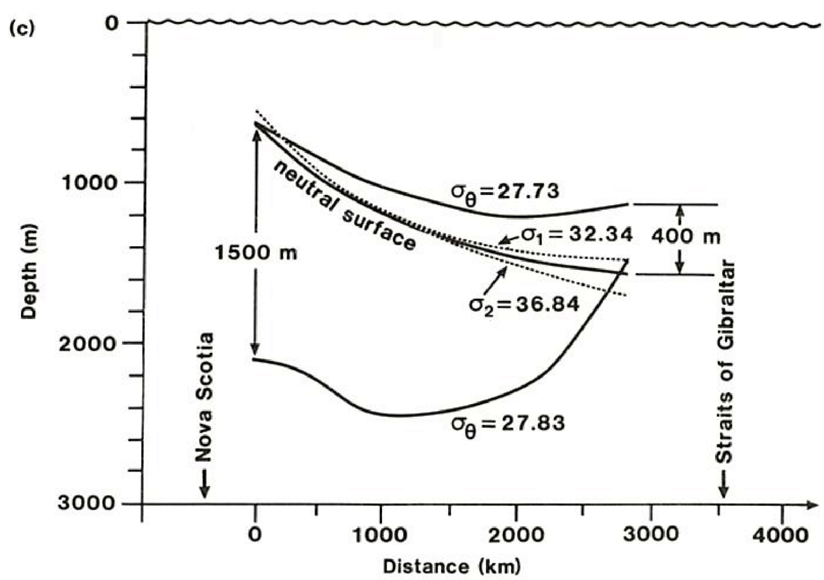
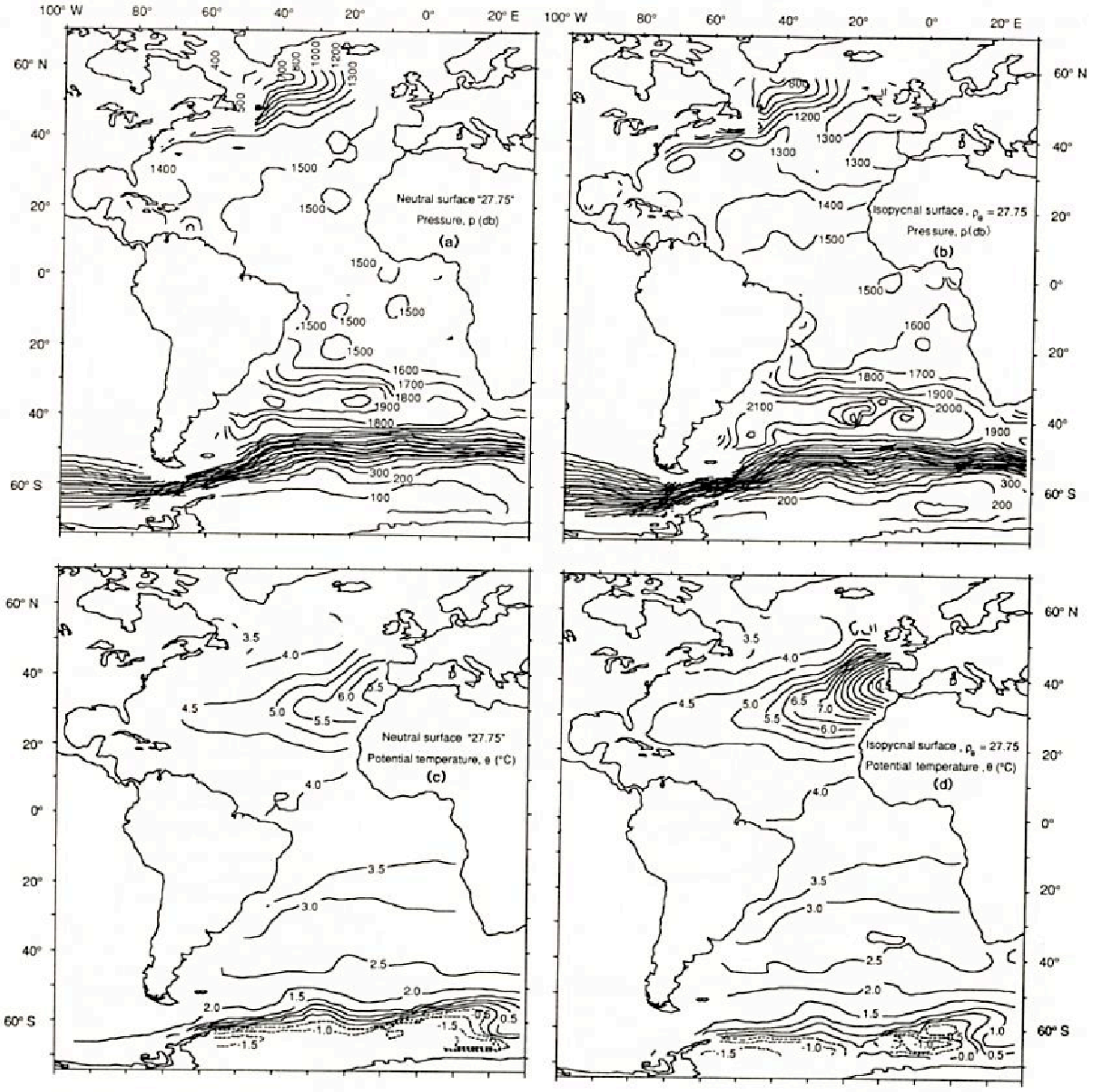


FIG. 7. Maps of pressure on two potential density surfaces: (a) σ_θ = 27.73; (b) σ_θ = 27.83. The potential density surfaces intersect the same neutral surface (NSa of Fig. 6) at different positions. This is illustrated in cross section in (c), which goes from near Nova Scotia on the left to near the Straits of Gibraltar on the right. Also shown (dashed lines) are a potential density surface referenced to a pressure of 1000 db (σ_1 = 32.34) and a potential density surface referenced to 2000 db (σ_2 = 36.84).



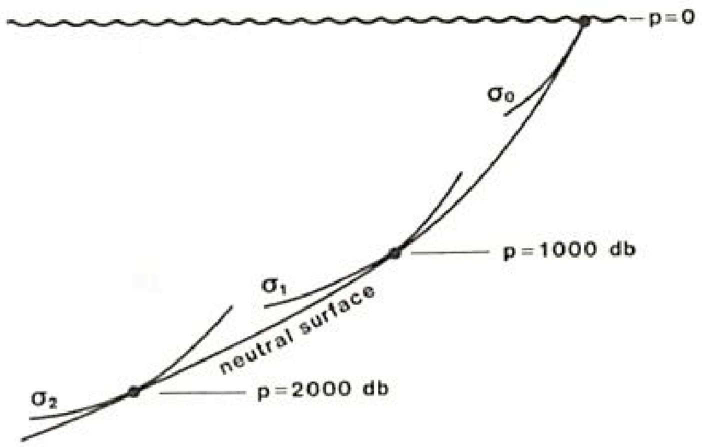
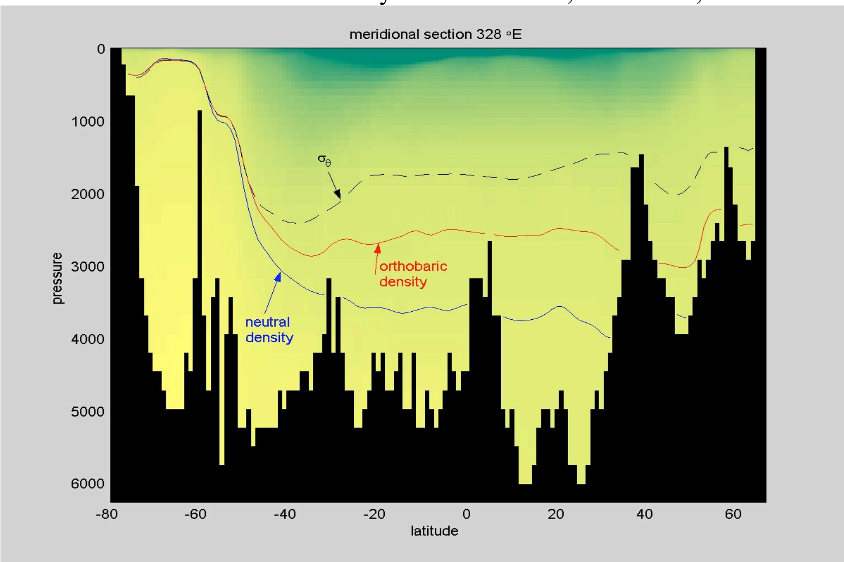
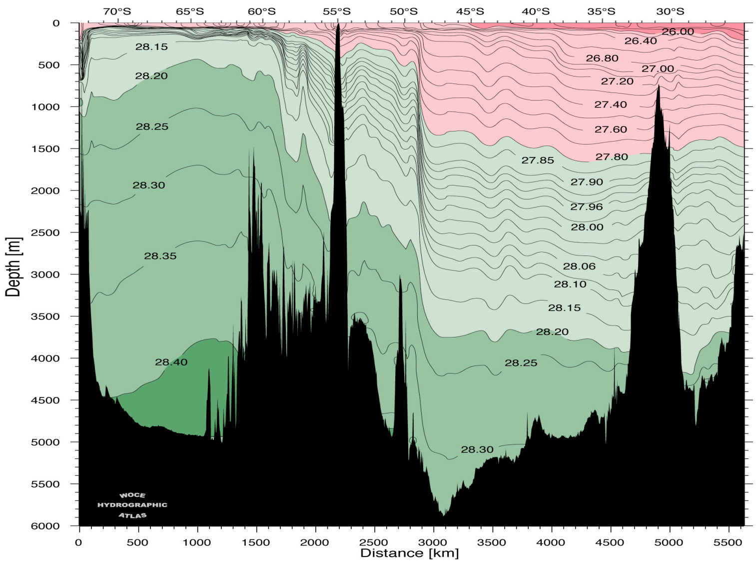
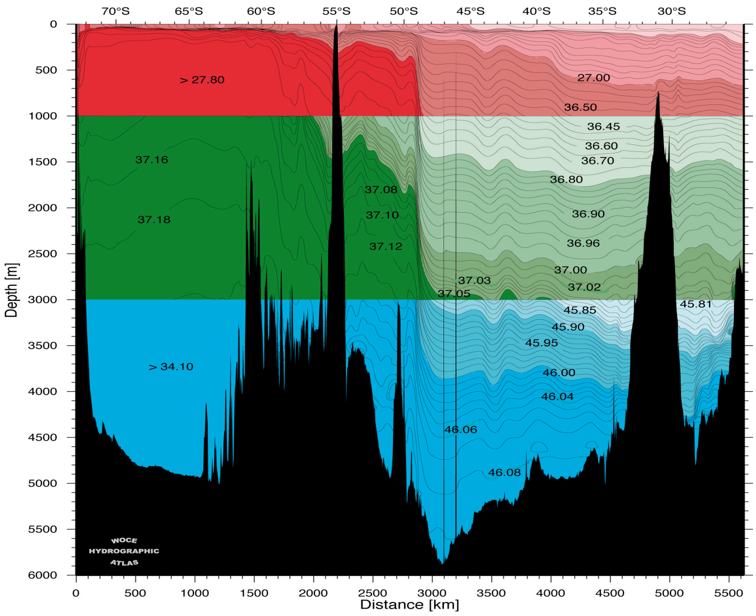


FIG. 2. Sketch of a neutral surface and three different potential density surfaces, referenced to 0 db, 1000 db and 2000 db. The neutral surface is tangential to potential density surfaces only at the reference pressure of those potential density surfaces. In this way, the neutral surface can be regarded as the envelope curve of many locally referenced potential density surfaces with continually changing reference pressures. The definition of a neutral surface adopted in this paper avoids the concept of potential density and in particular, avoids the changing reference pressure which is endemic to a neutral surface defined in terms of potential density concepts.



Below is a cross-section of Neutral Density in the Southern Ocean.





Before Neutral Density was available, cross-sections of density used potential density referenced to three different reference pressures, 0 dbar, 2000 dbar, and 4000 dbar, as shown above.

Geostrophic, hydrostatic and "thermal wind" equations

The geostrophic approximation to the horizontal momentum equations (Eqn. (B9)) equates the Coriolis term to the horizontal pressure gradient $\nabla_z P$ so that the geostrophic equation is

$$f\,\mathbf{k}\times\rho\mathbf{u} \;=\; -\nabla_z P \qquad \text{or} \qquad f\mathbf{v} \;=\; \tfrac{1}{\rho}\,\mathbf{k}\times\nabla_z P \;=\; g\,\mathbf{k}\times\nabla_p z\;, \qquad (3.12.1)$$

where $\mathbf{u}$ is the three dimensional velocity and $\mathbf{v} = -\mathbf{k}\times(\mathbf{k}\times\mathbf{u})$ is the horizontal velocity where $\mathbf{k}$ is the vertical unit vector (pointing upwards) and f is the Coriolis parameter. The last part of the above equation has used $\nabla_z P \;=\; -P_z\nabla_p z$ from Eqn. (3.12.4b) below and the hydrostatic approximation, which is the following approximation to the vertical momentum equation (B9),

$$P_z \;=\; -g\rho. \qquad (3.12.2)$$

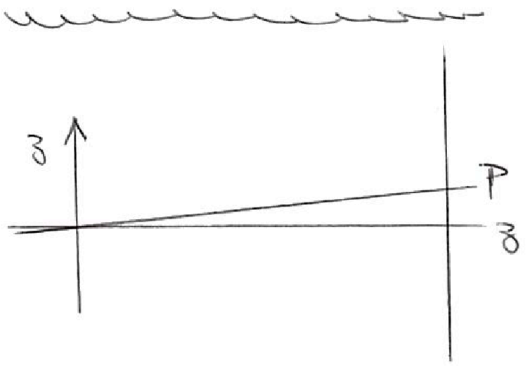
The use of P in these equations rather than p serves to remind us that in order to retain the usual units for height, density and the gravitational acceleration, pressure in these dynamical equations must be expressed in Pa not dbar.

The so called "thermal wind" equation is an equation for the vertical gradient of the horizontal velocity under the geostrophic approximation. Vertically differentiating Eqn. (3.12.1) and using the hydrostatic equation Eqn. (3.12.2), the thermal wind can be written

$$\boxed{f\,\mathbf{v}_z \;=\; \left(\tfrac{1}{\rho}\right)_z \mathbf{k}\times\nabla_z P + \tfrac{1}{\rho}\mathbf{k}\times\nabla_z\left(P_z\right) \;=\; -\tfrac{g}{\rho}\,\mathbf{k}\times\nabla_p\rho \;=\; \tfrac{N^2}{g\rho}\,\mathbf{k}\times\nabla_n P,} \qquad (3.12.3)$$

where ∇_p is the projected lateral gradient operator in the isobaric surface (see Eqn. (3.11.3)). The last part of this equation relates the "thermal wind", $f\,\mathbf{v}_z$, to the pressure gradient in the neutral tangent plane. Note that the Boussinesq approximation has not been made to derive any part of Eqn. (3.12.3). Under the Boussinesq approximation, $\nabla_p\rho$ is approximated by $\nabla_z\rho$, and the last term in Eqn. (3.12.3) is approximated as $-N^2\,\mathbf{k}\times\nabla_n z$. The derivation of Eqn. (3.12.3) proceeds as follows. To go from the second part of Eqn. (3.12.3) to the third part use is made of

$$\nabla_p\rho \;=\; \nabla_z\rho \;+\; \rho_z\nabla_p z \quad \text{and} \quad \nabla_p P \;=\; \mathbf{0} \;=\; \nabla_z P \;+\; P_z\nabla_p z\,. \qquad (3.12.4\text{a,b})$$



To go from the third part of Eqn. (3.12.3) to the final part, use is made of Eqn. (3.12.4a) and $\nabla_n\rho \;=\; \nabla_z\rho \;+\; \rho_z\nabla_n z$, which, when combined gives

$$\nabla_p\rho \;=\; \nabla_n\rho \;-\; \rho_z\left(\nabla_n z - \nabla_p z\right). \qquad (3.12.5)$$

Now Eqn. (3.12.4b) is used together with $\nabla_n P \;=\; \nabla_z P \;+\; P_z\nabla_n z$ to find

$$\nabla_n P \;=\; P_z\left(\nabla_n z - \nabla_p z\right), \qquad (3.12.6)$$

and this is substituted into Eqn. (3.12.5) to find

$$\nabla_p\rho \;=\; \nabla_n\rho \;-\; \rho_z\nabla_n P/P_z\,. \qquad (3.12.7)$$

Now along a neutral tangent plane we know that $\nabla_n\rho \;=\; \rho\kappa\nabla_n P$ (κ is the isentropic and isohaline compressibility of seawater) and substituting this into

Eqn. (3.12.7) leads to the final expression of Eqn. (3.12.3), namely $\frac{N^2}{g\rho}\,\mathbf{k}\times\nabla_n P$ (recognizing that the buoyancy frequency is defined by $N^2 = g\left(\kappa P_z - \frac{1}{\rho}\rho_z\right)$).